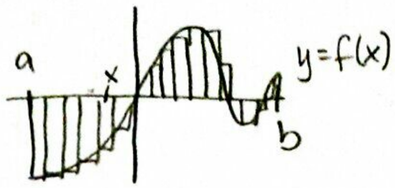
Lesson 31: The Fundamental Theorem of Calculus (FTC)

The FTC relates antiderivatives and definite integrals.

Notice



$$\int_a^b f(x)\,dx = \lim_{n\to\infty} L_n$$

↑ left Riemann sum (or R_n)

We can use this to prove FTC #1:

$$\frac{d}{dx}\left[\int_a^x f(t)\,dt\right] = f(x)$$

We'll only use FTC #2:

Theorem (FTC) If $f(x)$ is continuous on $[a,b]$ and $\int f(x)\,dx = F(x)$. ($F(x)$ is an antiderivative of $f(x)$)

then $\int_a^b f(x)\,dx = F(b) - F(a)$

$= F(x)\Big|_a^b \leftarrow$ notation

Proof If two functions have the same derivative ($F'(x) = G'(x)$), then $F(x) = G(x) + C$.

$$\underbrace{\frac{d}{dx}\left[\int_a^x f(t)\,dx\right] = f(x)}_{\text{FTC \#1}} \quad \text{and} \quad F'(x) = f(x)$$

So $\int_a^x f(t)\,dt = F(x) + C \quad \left(F(x) = \int_a^x f(t)\,dt - c\right)$

$$F(b) - F(a) = \left(\int_a^b f(t)\,dt - \cancel{c}\right) - \left(\int_a^a f(t)\,dt - \cancel{c}\right)$$

$$= \int_a^b f(t)\,dt - \cancel{\int_a^a f(t)\,dt} \to 0$$

$$= \int_a^b f(t)\,dt. \qquad \blacksquare$$

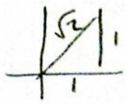
Ex 1 $\int_3^7 2\,dx = 2x\Big|_3^7 = 2(7) - 2(3) = 2(7-3) = 2\cdot 4 = \boxed{8}$

Ex 2 $\int_0^1 x^2\,dx = \frac{1}{3}x^3\Big|_0^1 = \frac{1}{3}(1)^3 - \frac{1}{3}(0)^3 = \boxed{\frac{1}{3}}$

Ex 3 $\int_0^{\pi/4} \tan x \cos x + 2\,dx = \int_0^{\pi/4} \underbrace{\cancel{\tan x}\,\cancel{\cos x}}_{\frac{\sin x}{\cancel{\cos x}}}\,dx + \int_0^{\pi/4} 2dx$

$= \int_0^{\pi/4} \sin x\,dx + \int_0^{\pi/4} 2\,dx$

$= -\cos x \Big|_0^{\pi/4} + 2x \Big|_0^{\pi/4}$

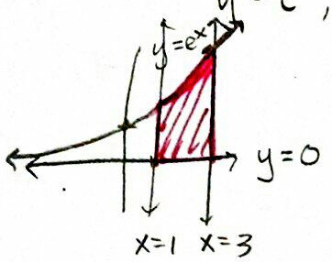


$= -\cos(\tfrac{\pi}{4}) - (-\cos(0)) + \cancel{2}(\tfrac{\pi}{\cancel{4}_{2}}) - 2(0)$

$= \boxed{-\frac{1}{\sqrt{2}} + 1 + \frac{\pi}{2}}$

$\frac{\sqrt{2}}{2}$

Ex 4 Find the area of the region bounded by $y = e^x$, $y = 0$, $x = 1$, and $x = 3$.



$\int_1^3 e^x dx = e^x \Big|_1^3 = e^3 - e^1 = \boxed{e^3 - e}$